Ultrametricity between states at different temperatures in spin-glasses

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Abstract. We prove the existence of correlations between the equilibrium states at different temperatures of the multi-p-spin spherical spin-glass models with continuous replica symmetry breaking: there is no chaos in temperature in these models. Furthermore, the overlaps satisfy ultrametric relations. As a consequence the Parisi tree is essentially the same at all temperatures with lower branches developing when lowering the temperature. We conjecture that the reference free energies of the clusters are also fixed at all temperatures as in the generalized random-energy model.

Introduction

The effects of temperature changes currently are the most interesting in spin-glass studies. The coexistence of chaos (or rejuvenation) and memory [2–4] is particularly puzzling. A first non-trivial problem is whether these effects are universal, *i.e.* whether they are present in different real spin-glasses [5] and in numerical simulations of microscopic models [6]; the main problem, however, lies in the theoretical explanation of these phenomena. If one admits that the static (equilibrium) free-energy landscape is somehow relevant to the off-equilibrium dynamics, the two effects of chaos and memory seem to contradict each other: the former points in the direction of absence of correlation between the equilibrium states at different temperature, while the latter points in the opposite direction. A qualitative explanation of the coexistence of the two effects has been given in terms of a hierarchical picture in which states at different temperatures are organized on the same ultrametric tree, whose details are revealed by lowering the temperature [7,8,3]. Recent numerical and theoretical work on the subject has confirmed that the generalized random energy model (GREM) [9,10], a model with such a landscape structure, displays chaos as well as memory [11]. However, the GREM is defined through its phase space, without any reference to any underlying microscopic description; therefore it would be interesting to find out whether there are spin-glasses that behave like the GREM. The question is whether equilibrium states at different temperatures are correlated or not and which is the nature of the correlations. The hypothesis of chaos in temperature affirms that they are uncorrelated in finite

dimensional systems [12,13]; this has been studied analytically in a number of paper [14–16]; We have questioned the hypothesis validity in the Sherrington-Kirkpatrick (SK) model in a recent paper [17], henceforth labeled [I]. The problem has also been investigated numerically [18–20]. It has also been shown that the random energy model (REM), which is non-chaotic by definition, is capable of producing strong rejuvenation signals [21]. On the other hand various attempts are underway to substantiate the idea of some kind of hierarchical structure underlying the phenomenology in the context of real-space theories like the droplet model [22,23].

In this paper we consider the mean-field multi-p-spin spherical spin-glass models for values of the coupling constants such that they display full replica symmetry breaking (RSB) [24]. It must be remarked that the analysis of the off-equilibrium dynamics [25] of these models has revealed the coexistence of chaos and memory [26]. We study the correlations between the equilibrium states at different temperatures following the lines developed in [I], where the problem has been put in connection with the existence of a particular class of solutions of the saddle point (SP) equations obtained in the replica framework. In Section 1 we prove that these solutions exist for the class of models considered, implying strong correlations between states at different temperatures, *i.e.* no chaos in temperature.

These solutions are built with Parisi matrices [1] and in Section 2 we show that this structure implies ultrametric relations between the overlaps of states at different temperatures. These ultrametric relations determine a one-to-one correspondence (except at the lower levels) between the trees of states at two different temperatures in such a way that to a cluster of states at temperature T_1, at a

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level of the T_1 tree labeled by some value q_1 of the overlap, corresponds a cluster of states at temperature T_2 at a level of the T_2 tree labeled by the same value $q_2 = q_1$. We call "twins" any two clusters in such a correspondence. Specifically, a state at temperature T_1 corresponds to a cluster of states at temperature $T_2 < T_1$ whose minimal overlap is given by q_{EA1}. Therefore each state at temperature T_1 can be considered as the father of many states at temperature T_2. A son has only one father, and it has an overlap lower than q_{EA1} with all the other states at T_1. The meaning of the correspondence between the trees at different temperatures is that any relationship between a state or cluster α at T_2 and any other state or cluster γ at T_1 or T_2 is univocally determined by the relationship between γ and the twin of α at T_1. In particular, given a state γ at T_2 and its father Γ at $T_1 > T_2$, the overlap between γ and any state Λ at T_1 is given by $q_{\Lambda\gamma} = q_{\Lambda\Gamma}$.

Each state of a spin-glass system at a given temperature has a certain statistical weight P_α determined by the Gibbs measure. According to the Parisi solution [1] the fluctuations of the weights with the disorder can be described through a stochastic process involving the free energies f_α of the states, which are defined as $P_\alpha = \exp[-\beta f_\alpha]/\sum_k \exp[-\beta f_k]$ [27,28]. Given the tree of states, one considers a given level k and assigns independently to each cluster of states at this level a reference free energy chosen randomly such that the average number of clusters with free energy between f and $f + \mathrm{d}f$ is given by

$$\mathrm{d}\mathcal{N}(f) = \exp[\beta x(q_k)f]\mathrm{d}f \tag{1}$$

where $x(q_k)$ is the inverse of the Parisi function $q(x)$. After having applied the procedure to every level of the tree, the free energy of a given state is set to be the sum of the reference free energies of the clusters to which it belongs at the various levels. For instance, if the state α belongs to the cluster i_1 at the first level of the tree, to the cluster $i_1 i_2$ at the second level of the tree and so on, its free energy is given by

$$f_\alpha = f_{i_1} + f_{i_1 i_2} + f_{i_1 i_2 i_3} + \ldots + f_{i_1 i_2 i_3 \ldots i_{L-1}} + f_{i_1 i_2 i_3 \ldots i_{L-1}\alpha}. \tag{2}$$

The distribution obtained has various interesting features; for instance, in each sample there are only few states with a finite weight, while there is an infinite number of states that carries an infinitesimal weight.

Since we found a one-to-one correspondence between the trees of states at different temperatures, it is natural to ask whether there are also correlation between the weights. In the replica framework the standard procedure to cope with this problem requires the computation of sums over all the different solutions of the saddle point equations, in order to obtain the cumulants of the distribution function of the weights. When considering a system at a given temperature, this can be done noticing that all the solutions are permutations of the standard Parisi solution [1,29]; therefore the sum over solutions can be replaced by a sum over replica indices. Unfortunately when considering systems at different temperatures the solutions are not permutations one of the other and we are unable to sum them

all. However, in Section 3 we conjecture that the weights distribution is identical to that of the GREM class of models. In the GREM the tree of states is the same at all temperatures by definition and the reference free energies of the clusters are constant too [9,10]. Their distribution obeys the law (1) with $\beta x(q)$ replaced by $y_u(q)$, a function that depends on the model. In analogy to this we guess that in the models we are considering the reference free energy of a given cluster at T_1 is equal to the reference free energy of its twin at T_2; we notice that since for this models we have $\beta x(q, T) = y_u(q)$, (where $y_u(q)$ depends on the coupling constants but not on the temperature), this is consistent with the fact that the same set of free energies obeys law (1) at all temperatures. This belief is motivated by the fact that there is a one-to-one correspondence between the solutions of SP equations of the multi-p spin spherical model and the solutions one finds in the analogous replica treatment of the GREM [16]. Indeed, the solutions have the same *formal* structure expressed in terms of the function $y_u(q)$ and the differences between the models show up only in the actual values of this function. Since for every solution of the multi-p spherical spin-glass models there is a correspondent solution of the GREM, we expect that the sums over solutions give equal results in the two models; more precisely we expect the results to have the same dependence on the function $y_u(q)$. Accordingly, the correlations between the weights at different temperatures will be equal in the two classes of models.

The previous conjecture could be proved if we were able to carry out the sum over all the solutions. This problem is common to many models where one would like to recover within the replica framework a result which is known *a priori* (*e.g.* in the REM [30]), or which can be obtained through different methods (*e.g.* the spherical p-spin model where the absence of chaos is easily seen within the TAP approach [35]). In Section 4 we discuss some technical difficulties connected with this problem. At the end we give our conclusions. Technical details are skipped to the appendices.

1 Correlations between states at different temperatures

We recall the method discussed in [I] to study the correlations between states at different temperatures. The replica trick is usually used in order to compute the average over the disorder of the free energy by computing the moments of the partition function $\overline{Z^n}$. In the thermodynamic limit saddle point (SP) equations are obtained for the order parameter which is an $n \times n$ matrix Q_{ab}. An outcome of the computation is that the order parameter is connected to the distribution of the overlaps through the relation

$$q^{(k)} = \int q^k P(q)\mathrm{d}q = \lim_{n \to 0} \sum_{\text{all solutions}} Q_{ab}^k. \tag{3}$$

Where the function $P(q)$ is the averaged probability density of finding two states with overlap q according to their

Gibbs weight, and a and b are two different replica indices (*e.g.* 1 and 2). In the r.h.s. of (3) there appears a sum over all the different solutions of the saddle point equation with the same (maximum) free energy; indeed, as soon as replica symmetry is broken, we have many solutions: given a certain solution others can be obtained through a permutation of the replica indices. However, in the isothermal case, all the solutions are given by the Parisi solution plus its natural permutations; therefore the sum over them can be replaced with an average over the replica indices multiplied by the total number of different solutions, which in the $n \to 0$ limit goes to 1 [1,29]. In [I] it has been shown that the correlations between states at different temperatures can be investigated through the computation of the $n \to 0$ limit of the quantity $(1 - \overline{Z^n(T_1)Z^n(T_2)})/n$. As a matter of fact, we are not interested in its actual value (which must be equal to the sum of the free energies at temperature T_1 and T_2), but in the saddle point equation obtained by applying the replica trick to the whole $\overline{Z^n(T_1)Z^n(T_2)}$ and not to $\overline{Z^n(T_1)}$ and $\overline{Z^n(T_2)}$ separately[1]. In this case one obtains SP equations for a $2n \times 2n$ matrix $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$, where Q_1, Q_2 and P are $n \times n$ matrices. The advantage of this procedure is that the matrix P encodes information on the correlations between states at different temperatures; indeed, through the same steps that led to (3) it can be shown that

$$q_{T1T2}^{(k)} = \int q_{T1T2}^k P(q_{T1T2}) \mathrm{d}q_{T1T2} = \lim_{n \to 0} \sum_{\text{all solutions}} P_{ab}^k.$$

Where the function $P(q_{T1T2})$ is the generalization of the $P(q)$ to states at different temperatures. As above, we have to sum over all the solutions of the SP equations: this is quite a delicate point; indeed we do not expect that all solutions be natural permutations of a single solution as in the isothermal case (*i.e.* the standard Parisi solution). Actually, there are infinite solutions corresponding to different parameterizations and we don't know how to sum them all. However, it is possible to reconstruct the function $P(q_{T1T2})$ in an indirect way by knowing the weight distribution of the equilibrium states. We have applied the method to the spherical spin-glass model with multi-p-spin interaction [24]. The Hamiltonian of the model is defined as

$$H = \sum_{p=2}^{\infty} \sum_{i_1 < i_2 < \ldots < i_p} J_{i_1 i_2 \ldots i_p} S_{i_1} S_{i_2} \ldots S_{i_p} + h \sum_i S_i.$$

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[1] This quantity is actually the logarithm of the partition function of the two systems. To make the text more readable we will refer to it as the "free energy" of the two systems, while the free energy of a single system is the logarithm of its partition function *multiplied* by β, the inverse of its temperature. Therefore a claim like "the free energy of the two systems must be equal to the sum of the free energies at temperature T_1 and T_2" is shorthand for "the logarithm of the partition function of the two systems must be equal to the sum of the logarithms of the partition functions at temperature T_1 and T_2".

The J's are independent Gaussian random variable with zero mean and variance $\left\langle J_{i_1 i_2 \ldots i_p}^2 \right\rangle = (p-1)! J_p^2 N^{1-p}$. The spins are subjected to the spherical constraint $\sum_i S_i^2 = N\sigma$. Introducing the function

$$f(q) = \sum_{p=2}^{\infty} \frac{1}{p} J_p^2 q^p.$$

The replicated free energy reads

$$2\beta F_n = -\beta^2 \sum_{ab} [f(Q_{ab}) + H^2 Q_{ab}] - \mathrm{Tr} \ln Q - n$$

where the variational parameter is an $n \times n$ matrix Q_{ab} with diagonal $q_d = \sigma$. The SP equations then read

$$\beta^2 f'(Q_{ab}) + \beta^2 H^2 = -\left(\frac{1}{Q}\right)_{ab}. \qquad (4)$$

Expressing Q_{ab} as a Parisi function we obtain the solution as $q(x, T) = q_u(\beta x)$ where $q_u(y)$ is defined as the inverse of the universal function

$$y_u(q) = \frac{f'''(q)}{2f''(q)^{3/2}}$$

with the value of $q(1)$ fixed by

$$q_d - q(1) = \frac{1}{\beta \sqrt{f''(q(1))}}.$$

In the presence of a small magnetic field there is also a small plateau of temperature-independent height q_H fixed by the condition

$$H^2 = q_H f''(q_H) - f'(q_H).$$

When we consider simultaneously two systems at different temperature, we get the following expression for $\overline{Z^n(T_1)Z^n(T_2)}$ in the thermodynamic limit

$$F_n = \ln \overline{Z^n(T_1)Z^n(T_2)} = -\beta_1^2 \sum_{ab} (f(Q_{1ab}) + H^2 Q_{1ab})$$
$$- \beta_2^2 \sum_{ab} (f(Q_{2ab}) + H^2 Q_{2ab}) +$$
$$- 2\beta_1 \beta_2 \sum_{ab} (f(P_{ab}) + H^2 P_{ab}) - \mathrm{Tr} \ln \hat{Q} - n. \quad (5)$$

The corresponding SP equations are similar to those for the isothermal case (4) and are reported in Appendix B. A solution of the SP equations is certainly the one with $P = 0$ and Q_1 and Q_2 equal to the corresponding isothermal solutions. Its free energy is given by the sum of the free energies at temperatures T_1 and T_2, as expected. The problem is whether other solutions exist with a non-zero P and with the same free energy of the $P = 0$ solution. In [I] a particular structure was proposed for these $P \neq 0$ solutions (called non-chaotic, since their existence implies absence of chaos in temperature); we found that solutions
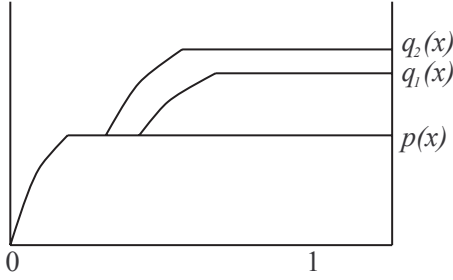
Fig. 1. The solutions for $p_d \neq 0$. In the small-x region they are all equal to $q_u((\beta_1 + \beta_2)x)$ until the point x_c where $p(x) = p_d$, for $x > x_c$ $p(x)$ is constant and equal to p_d, while $q_1(x)$ and $q_2(x)$ after an intermediate plateau are joined respectively to $q_u(\beta_1 x)$ and $q_u(\beta_2 x)$; p_d can take values between zero and q_{1EA}.

having such a structure do exist for this class of models. Actually there is an infinite number of solutions which are parameterized by the value p_d of the diagonal of P, whose value ranges from zero to a maximum one, which, for these models, turns out to be the self-overlap of the states at the higher temperature. The solution with $P = 0$ is included in this set and corresponds to the value $p_d = 0$. It is very interesting to notice that they exist also when the two systems at different temperatures are subjected to the same magnetic field (see Appendix B). Here for simplicity we will refer to the $H = 0$ case. For a given value of p_d the solutions in terms of the three functions $q_1(x)$, $q_2(x)$ and $p(x)$ are

$$q_s(x) = \begin{cases} q_u((\beta_1 + \beta_2)x) & \text{for} \quad x \leq \frac{1}{\beta_1 + \beta_2} y_u(p_d) \\ p_d & \text{for} \quad \frac{1}{\beta_1 + \beta_2} y_u(p_d) \leq x \leq \frac{1}{\beta_s} y_u(p_d) \\ q_u(\beta_s x) & \text{for} \quad \frac{1}{\beta_s} y_u(p_d) \leq x \leq x_{max}(T_s) \end{cases}$$

(6)

$$p(x) = \begin{cases} q_u((\beta_1 + \beta_2)x) & \text{for} \quad x \leq \frac{1}{\beta_1 + \beta_2} y_u(p_d) \\ p_d & \text{for} \quad \frac{1}{\beta_1 + \beta_2} y_u(p_d) \leq x \leq 1 \end{cases}$$

(7)

where $q_u(y)$ and $y_u(q)$ are the universal function defined above. These solutions can be built for any couple of temperatures $T_1 \geq T_2$ both below the critical temperature and for any p_d between zero and $q_1(1)$ which is the self-overlap of the states of the system at the higher temperature. The solutions are sketched in Figures 1 and 2. Notice that $q_1(x)$, $q_2(x)$ and $p(x)$ are all equal in the small-x region. This is connected with the fact that this class of models verifies the scaling $q(x, T) = q(x/T)$. We skip to Appendix B the demonstration that these functions solve the SP equations and have the correct free energy. It must be remarked that the solutions (6, 7) are formally the same which have been found with a similar treatment of the GREM [16], with a different model-dependent function $y_u(q)$. At this stage we can infer that the function $P(q_{T1T2})$ has a non-zero support from zero to the self-overlap of the states at the higher temperature.
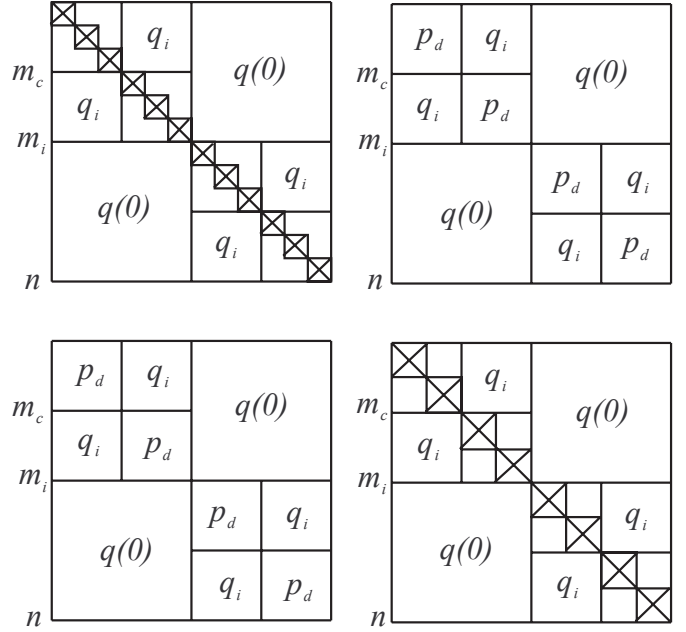

Fig. 2. The global matrix $\hat{Q}$ for a given p_d. Q_1 is at the left high corner, Q_2 at the low right corner, P at the left low corner. If we consider two replicas a and b at temperature T_1 such that $Q_{1ab} \geq p_d$ the overlaps P_{ac} and P_{cb} with a replica c at temperature T_2 verify $P_{ac} = P_{bc} \leq p_d$; if $Q_{1ab} \leq p_d$ then we may have $P_{ca} = Q_{1ab} \leq P_{cb}$, or $P_{cb} = Q_{1ab} \leq P_{ca}$, or $P_{cb} = P_{ca} \leq Q_{1ab}$; *i.e.* we have ultrametric relations between replicas at different temperatures.

2 Ultrametricity

The solutions we found imply ultrametricity between states at different temperatures of a given system. To see this we must consider the probability $P(q_{12}, q_{13}, q_{23})$ of extracting three states at different temperatures with assigned values of their mutual overlap. If we assume, for instance, that states 1 and 2 are at temperature T_1 and state 3 is at temperature T_2, this function is related to the solutions of the SP equation through the following relation

$$\int q_{12}^r q_{13}^s q_{23}^t P(q_{12}, q_{13}, q_{23}) \mathrm{d}q_{12} \mathrm{d}q_{13} \mathrm{d}q_{23} =$$

$$\lim_{n \to 0} \sum_{\text{all solutions}} Q_{1ab}^r P_{ac}^s P_{cb}^t.$$

We are not able to perform the sum over solutions in r.h.s of the previous expression, but we can infer ultrametricity by simply looking at their structure. The function $P(q_{12}, q_{13}, q_{23})$ can be reconstructed later in an indirect way from the distribution of the weights we shall describe below (Sect. 3).

Let us consider the solution for a given p_d: a simple analysis of Figure 2 shows that for any three replica indices a, b, and c, the corresponding overlaps Q_{1ab}, P_{ac} and P_{cb} always form an isosceles triangle with the two equal sides smaller or equal to the third. However, while Q_{1ab} can take values from zero to q_{EA1}, P_{ac} and P_{cb} can take values from zero to p_d. Considering other solutions, we

$$\begin{aligned}
M_{kj} &= \overline{\sum_{a_1 \ldots a_k b_1 \ldots b_j} P_{a_1} \ldots P_{a_k} P_{b_1} \ldots P_{b_j} \Theta(q_{a_1 a_2} - q) \ldots \Theta(q_{a_1 a_k} - q)\Theta(q_{a_1 b_1} - q) \ldots \Theta(q_{a_1 b_j} - q)} \\
&= \lim_{n \to 0} \sum_{\text{all solutions}} \Theta(Q_{1a_1 a_2} - q) \ldots \Theta(Q_{1a_1 a_k} - q)\Theta(P_{a_1 b_1} - q) \ldots \Theta(P_{a_1 b_j} - q).
\end{aligned}$$

obtain values of P between 0 and q_{1EA}. Now, since ultrametricity is a property of all the solutions, it will be a property of the sum over solutions as well; therefore the overlaps between any three states at any temperature satisfy ultrametric relations. In general, the maximum overlap between a state at temperature T_1 and another state at temperature T_2 is equal to the self-overlap of the state at the higher temperature. Perfect ultrametricity between states at different temperatures is a special property of this class of models which relies on the identity of the three functions $q_1(x)$, $q_2(x)$ and $p(x)$ in the small-x region.

As we said in the introduction, the ultrametric relations between the overlaps of states at different temperatures define a correspondence between the trees of states. Consider a cluster I_1 of equilibrium states at temperature T_1 whose overlaps are greater or equal to some q_1. Given a state α in I_1, we have that the overlap between any other state β in I_1 and any state γ outside I_1 is simply given by $q_{\beta\gamma} = q_{\alpha\gamma}$ [1]. Now, given a state α' at T_2 whose overlap with any of the states in I_1 is greater or equal to q_1, we have that the overlap between α' and any state γ at T_1 outside I_1 is simply given by $q_{\alpha'\gamma} = q_{\alpha\gamma}$; In other words, from a geometrical point of view we may assume that α' *is* in I_1. Furthermore, this is true for all the states at T_2 whose overlap with α' is greater or equal than q_1; by definition they form a cluster I_2 which we call the "twin" of I_1. As far as the overlaps are concerned the two clusters I_1 and I_2 can be considered the same; therefore from a topological point of view there is only one tree of states whose details are revealed by lowering the temperature. Finally, we recall that the solutions we described satisfy the separability property [38,17], which means that the overlap contains all the information between two states at equal or different temperatures.

3 The distribution of the weights

Each state of a spin-glass system at a given temperature has a certain statistical weight P_α. The fluctuation of the weights with the disorder can be described through the following procedure [1,27,28] involving the free energies of the states defined as $P_\alpha = \exp[-\beta f_\alpha]/\sum_k \exp[-\beta f_k]$. Given the tree of states, we consider a given level and assign to each cluster of states at this level a reference free energy chosen randomly such that the average number of clusters reference with free energy between f and $f + df$ is given by

$$d\mathcal{N}(f) = \exp[\beta x(q_k)f]df.$$

After applying the procedure to any level of the tree we define the free energy of a given state as the sum of the reference free energies of the cluster to which it belongs at the various levels. For instance, if the state α belongs to the cluster i_1 at the first level of the tree, to the cluster $i_1 i_2$ at the second level of the tree and so on, its free energy is given by

$$\begin{aligned}
f_\alpha = f_{i_1} + f_{i_1 i_2} + f_{i_1 i_2 i_3} + \ldots \\
+ f_{i_1 i_2 i_3 \ldots i_{L-1}} + f_{i_1 i_2 i_3 \ldots i_{L-1}\alpha}.
\end{aligned} \tag{8}$$

We have established that the trees of states at different temperatures are equal: to a cluster of states at T_1 corresponds a cluster of states at T_2; therefore we want to know which is the relation between the corresponding reference free energies. This correlation can be obtained computing the quantities

$$M_{kj} = \overline{\sum_I W_{I,1}^k W_{I,2}^j}.$$

Where the index I refers to the clusters at a certain level q of the tree and $W_{I,1}$ and $W_{I,2}$ are the weight of the cluster I respectively at temperature T_1 and T_2. Through standard manipulation [1] we obtain

See equation above.

Again, the problem is reduced to the computation of a sum over all the solutions of the SP equations. We are unable to perform such a sum but we have a line of reasoning to guess which may be the result. Indeed, the previous expression does not depend on the model under consideration; it is a general outcome of the replica trick. Therefore, if we have two different models with the same set of solutions the corresponding weight distribution functions will be equal. The solutions (6,7) found above are formally the same which have been found with a similar treatment of the GREM [16], with a scaling $q_1(x) = q_2(x) = p(x) = q_u((\beta_1 + \beta_2)x)$ in the small-x region. In other words, for any solution with $P \neq 0$ of the spherical model with multi-spin interactions there is a corresponding solution of the GREM with the same parameterization. Since there is a one-to-one correspondence between solutions of the GREM and solutions of the multi-p-spin spherical model, we expect all the quantities of interest to have the same formal dependence on the temperatures and on the universal function $q_u(y)$. This correspondence prompt us to conjecture that the structure of the correlations between the weights is the same in the two models, *i.e.* that *the reference free energy of a given cluster at temperature T_1 is equal to the reference free energy of the corresponding cluster at temperature T_2.* To be

$$q_{11}(x) = q_{22}(x) = q_{33}(x) = q_{12}(x) = p_{13}(x) = p_{23}(x) = \begin{cases} 0 \text{ for } x < 1/(2\beta_1 + \beta_2) \\ 1 \text{ for } x > 1/(2\beta_1 + \beta_2) \end{cases}$$

$$q_{44}(x) = \begin{cases} 0 \text{ for } x < 1/\beta_2 \\ 1 \text{ for } x > 1/\beta_2 \end{cases}$$

$$p_{14}(x) = p_{24}(x) = q_{34}(x) = 0. \tag{10}$$

precise one should notice that the $q(x)$ of the GREM has a qualitatively different shape from that of the models we are considering; indeed the solutions of the GREM does not have a plateau at q_{max} and furthermore $q(x) = 0$ for $x < T/T_c$. This makes no difference at all: since the solutions has the same formal expression in terms of the function $y_u(q)$, the formal expressions of quantities like the $P(q_{T1T2})$ must be the same, and the difference between the two models is only in the actual value of $y_u(q)$.

According to the previous argument the reference free energies of the various clusters are fixed at all temperatures, in particular the ordering of the clusters according to them does not change; however, this does not mean that the ordering of the clusters according to their actual weights is also conserved: this is only true on average. In other words, it is not true that to the heaviest cluster at a given temperature corresponds the heaviest cluster at a lower temperature.

4 Solutions with different parameterizations

In the previous sections we saw that in order to obtain the quantities of interest we should perform sums over all the different solutions of the SP equations of the functional (5). Here we intend to discuss one of the main difficulties which prevents us from performing such sums, *i.e.* the existence of an infinite set of solutions corresponding to different parameterizations of the $2n \times 2n$ order parameter $\hat{Q}$.

By applying the same technique of Section 2 to the REM [30] we found the same essential features of the problem. Here the solution of the standard isothermal problem is a very simple $q(x)$ whose value is 0 for $x < 1/\beta$ and 1 for $x > 1/\beta$; this is the simplest case of a solution that verifies the so-called Parisi-Toulouse scaling $q(x,T) = q_u(\beta x)$ [31], which is also found in models with full RSB like those we are considering.

As we saw in Section 2 the problem is connected to the extremization of the quantity $\ln \overline{Z^n(T_1)Z^n(T_2)}$ with respect to the order parameter which is a $2n \times 2n$ matrix $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$, where Q_1, Q_2 and P are $n \times n$ matrices. One can show that there exist solutions of the type (6,7) for the REM; in particular, we have that Q_1, Q_2 and P are 1RSB matrices with values 0 for $x < 1/(\beta_1 + \beta_2)$ and 1 for $x > 1/(\beta_1 + \beta_2)$.

In the solutions (6,7) the matrices Q_1, Q_2 and P are Parisi matrices, but there exist other solutions corresponding to different parameterizations. For instance we can

divide each of the three matrices Q_1, Q_2 and P in four blocks of size $n/2 \times n/2$. We have then 16 blocks and we can parameterize each of them as a Parisi matrix. In the $n \to 0$ limit the corresponding SP equations are the same obtained by considering four different system, two of them at T_1 and the other two at T_2. The order parameter in the latter case is a $4n \times 4n$ matrix composed of 16 matrices of size $n \times n$

$$\hat{Q} = \begin{pmatrix} Q_{11} & Q_{12} & P_{13} & P_{14} \\ Q_{12} & Q_{22} & P_{23} & P_{24} \\ P_{13} & P_{23} & Q_{33} & Q_{34} \\ P_{14} & P_{24} & Q_{34} & Q_{44} \end{pmatrix}. \tag{9}$$

When we parameterize each of these matrices through Parisi functions $q_{11}(x), q_{12}(x)\ldots$ and take the limit $n \to 0$ the corresponding equations are identical as they would be if the size of the matrices were $n/2$ rather than n; in other words, the solutions of the four-system problem are also solutions of the two-system problem, and they offer another way of parameterizing the order parameter $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$. This argument can be extended indefinitely considering at the same time a general number of $2p$ systems, p at temperature T_1 and p at temperature T_2. In this way an infinite set of possible parameterizations of the $2n \times 2n$ order parameter $\hat{Q}$ in terms of Parisi matrices is obtained.

However, not all the parameterizations give new solutions. Going back to parameterization (9) we have 16 Parisi matrices to consider; in the REM we have a solution in which they are all equal 1RSB matrices with a breaking point $x_c = 1/(2\beta_1 + 2\beta_2)$. It can be seen that this is nothing but a permutation of the solution in which Q_1, Q_2 and P are 1RSB matrices with $x_c = 1/(\beta_1 + \beta_2)$ (see also the discussion in the Conclusions). Therefore the 4-system parameterization does not add anything new in this case. However, the 4-system parameterization describes also the following solution

See equation (10) above.

This solution cannot be obtained through a permutation from the solution of the type (6,7) and should be counted separately. Essentially, this parameterization corresponds to two systems (system 1 and 2) at temperature T_1 correlated with a system at temperature T_2 (system 3), while system 4 at T_2 is completely uncorrelated to the remaining three. Accordingly, in summing over all the different solutions we must consider the infinite set of solutions whose parameterization corresponds to a generic number of m_1 systems at temperature T_1 correlated to a generic

number of m_2 systems at temperature T_2, plus $m_2 - m_1$ (if $m_2 > m_1$) uncorrelated systems at temperature T_1 (in order to have the same number of system at each temperature). The scaling of the correlated components is $q(x, T_1, T_2) = q_u((m_1\beta_1 + m_2\beta_2)x)$, *i.e.* in the REM the correlated components are described by 1RSB function with $x_c = 1/(m_1\beta_1 + m_2\beta_2)$.

According to [I] any solution with a given parameterization can also be considered as solution of a certain constrained system. For instance, the blocks $Q_{11}, Q_{22}, Q_{33}, Q_{12}, P_{13}, P_{23}$, in the solution (10) correspond to a system of three real replicas which are constrained to have maximum overlap (*i.e.* 1 in the REM) among themselves. When considering a constrained system the problem of summing over all solutions greatly simplifies; indeed in this case we have only one solution plus its natural permutations and we can perform the sum in the standard way by replacing it with a sum over indices. The result is immediate, for the $P(q)$ of the constrained system we have $P(q) = \delta(q)/(2\beta_1 + \beta_2) + \delta(q-1)(1 - 1/(2\beta_1 + \beta_2))$. We define the "free energy" f_α of a state of the global system (composed of two constrained systems) as $P_\alpha = \exp[-f_\alpha]/\sum_\beta \exp[-f_\beta]$ (see footnote 1). Then we obtain that these free energies are independent random variables such that the average number of states with free energies between f and $f + \mathrm{d}f$ is $\mathrm{d}\mathcal{N}(f) = \exp[f/(2\beta_1 + \beta_2)]\mathrm{d}f$. This result is precisely what is obtained noticing that the states of the constrained system are triplets of identical states of the single system; therefore in the REM their free energy is simply given by $f_\alpha^{\text{constrained}} = (2\beta_1 + \beta_2)f_\alpha^{\text{single}}$ (see footnote 1) and the distribution of f^{single} is proportional to $\exp[\beta_1 x_1 f] = \exp[\beta_2 x_2 f] = \exp[f]$. The same line of reasoning applies to the models we considered in the previous sections as well: it ensures that the results obtained by considering constrained systems (the only case in which we are able to sum over the solutions explicitly) are fully consistent with our guess on the distribution of the free energies made in Section 3.

5 Conclusions

We proved the existence of correlations between the equilibrium states at different temperatures of the spherical spin-glass models with multi-p spin interactions for values of the coupling constants such that they display full RSB: there is no chaos in temperature in such models. Furthermore, the overlaps between states at different temperatures satisfy ultrametric relations. Ultrametricity determines a one-to-one correspondence between the trees of states at different temperature: to any cluster of states at temperature T_1 at a given level of the tree corresponds a "twin" cluster of states at temperature T_2; the precise meaning of this correspondence has been discussed in details in the introduction and in Section 2. From a purely geometrical point of view we may say that there is only one tree of states at all temperatures, whose details are revealed lowering the temperature. Prompted by some technical features of the problem, we conjectured that the reference free energies of two twins clusters are equal at all temperatures, exactly as in the GREM. Consequently,

we expect that quantities like the $P(q_{T_1T_2})$ in these models are the same as the GREM. More precisely we expect them to have the same formal dependence on the temperatures and on the function $y_u(q)$, whose actual values will be different in the two classes of models.

The scenario of a tree of states which bifurcates when lowering the temperature was suggested early in spin-glass studies [32] and was later advocated in order to explain the phenomenology of rejuvenation and memory in spin-glasses [8]. According to our findings, the basic premises of that phenomenological picture of off-equilibrium dynamics hold for the class of models considered here; we believe that this further increases the need for a complete understanding of the relationship between the equilibrium energy landscape and the off-equilibrium dynamics in spin-glasses.

Within the TAP approach [1,33] strong correlations between the equilibrium states at different temperatures are readily obtained in the spherical p-spin model with 1RSB [34–36]. Indeed in this model the angular and the self-overlap contributions to the TAP free energy can be factored; that is if we write $m_i = q^{1/2}\widehat{m_i}$, we have that the angular part $\widehat{m_i}$ enters the expression of the TAP free energy only in the form $E_p = q^{p/2} \sum J_{i_1 i_2 \ldots i_p} \hat{m}_{i_1} \hat{m}_{i_2} \ldots \hat{m}_{i_p}$. Accordingly, the angular components of the states, which are the extrema of the free energy, are the same at all temperatures. To our knowledge this was the only result on the absence of chaos previously obtained for a spin-glass model. The previous argument cannot be used in dealing with spherical spin-glass models with multi-p interactions for values of the coupling constants such that they display either 1RSB or full-RSB. For instance, in the $q^2 - q^4$ model there is an angular dependence of the form $E = q \sum J_{ij} \hat{m}_i \hat{m}_j + q^2 \sum J_{ijkl} \hat{m}_i \hat{m}_j \hat{m}_k \hat{m}_l$ and no factorization is possible. Indeed, when the temperature changes the self-overlap changes with it causing the angular landscapes corresponding to the two interactions to interpenetrate. It would be interesting to understand how this process leads to the properties we found within the replica approach. Furthermore, we recall that the off-equilibrium dynamics [25] is connected to the TAP free-energy landscape [40].

As discussed in [I], the solutions (6, 7) have the same free energy of the $P = 0$ solution. This can be also derived noticing that they form a continuous line in the space of the matrices $\hat{Q}$ parameterized by the value of the continuous parameter p_d. The value $p_d = 0$ corresponds to the standard Parisi solutions at the two temperatures. On this continuous line of solutions we have $\partial F/\partial \hat{Q} = 0$ by definition; therefore the free energy is constant and equal to that of the $p_d = 0$ solution. The previous argument has deep consequences. It provides an easy way to understand the Goldstone Theorem for disordered systems with full RSB which has been proved and discussed in [39]. This theorem connects the presence of Goldstone modes in spin-glasses with continuous RSB with the fact that the discrete permutational symmetry within the replica approach becomes a continuous symmetry when the number of RSB steps goes to infinity.

Consider the standard replica formulation of a given model: it deals with a $n \times n$ order parameter Q_{ab}. Instead of making the Parisi ansatz on it, we can divide it in four $n/2 \times n/2$ blocks and make the Parisi ansatz on each of these blocks separately, *i.e.* we assume that the two diagonal blocks are equal to an ultrametric $n/2 \times n/2$ matrix q_{ab} and the two off-diagonal blocks are equal to a $n/2 \times n/2$ ultrametric matrix p_{ab}. Then the SP equations are written in terms of two functions $q(x)$ and $p(x)$ defined on the interval $[1, n/2]$, but in the $n \to 0$ limit the equations are identical to those obtained considering $2n$ replicas with an order parameter $\hat{Q}$ parameterized by four $n \times n$ matrices, the two diagonal ones equal to Q_{ab} and the two off-diagonal ones equal to P_{ab}. This is a special case ($T_1 = T_2$) of the problem we considered in this paper. In this case [37] the solutions corresponding to (6, 7) read

$$q(x) = p(x) = q_{Parisi}(2x) \qquad 0 \le x \le \frac{1}{2}x_{Parisi}(p_d)$$

$$q(x) = p(x) = p_d \frac{1}{2}x_{Parisi} \qquad (p_d) \le x \le x_{Parisi}(p_d)$$

$$q(x) = q_{Parisi}(x); \qquad p(x) = p_d x_{Parisi}(p_d) \le x \le 1 \tag{11}$$

where $q_{Parisi}(x)$ is the standard Parisi solution and p_d is a continuous parameter with range $[0, q_{EA}]$; every value of p_d specifies a solution. As stated above, these solutions, first obtained considering $2n$ replicas, solve the standard n-replica problem too. At this point we can make the following statements:

– These solutions exist in every model with Parisi RSB, either discrete of continuous. This statement has already appeared in [16] where it is claimed that it is implied by ultrametricity. We justify this result noticing that these solutions are *permutations* of the standard Parisi solution, as it is readily understood considering Figure 3.

– When the model has full RSB the parameter p_d is continuous, therefore the solutions form a continuous line of constant free energy. As a consequences on every point of the line the Hessian has zero eigenvalues.

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Appendix A

In this appendix we derive the formulas to compute a generic function (*e.g.* inverse, logarithm...) of one or more Parisi matrices. Using the standard eigenvalue technique one encounters some difficulties. The main problem is that in the $n \to 0$ limit the matrix ceases to be determined univocally by the set of its eigenvalues. For a generic Parisi matrix parameterized as $(a_d, a(x))$ the eigenvalues are [34]

$$\lambda_a(0) = a_d - \int_0^1 a(y)\mathrm{d}y \qquad \deg : 1 \tag{12}$$

$$\lambda_a(x) = a_d - xa(x) - \int_x^1 a(y)\mathrm{d}y \; \deg : -n\frac{\mathrm{d}x}{x^2}. \tag{13}$$
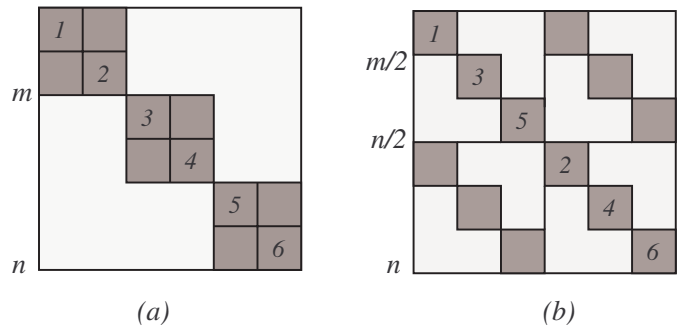

Fig. 3. (a) The solution of a generic spin-glass problem with 1RSB. (b) A permutation of the solution (a) which corresponds to a parameterization of the $n \times n$ order parameter as four $n/2 \times n/2$ Parisi matrices. The generalization to full RSB is straightforward. This figure shows that the solutions (11) are obtained from a permutation of the standard Parisi solution $q_{Parisi}(x)$, in particular the origin of the small-x scaling $q(x) = p(x) = q_{Parisi}(2x)$ is clarified.

Two important eigenvalues are $\lambda_a(0) = a_d - \overline{a}$ and $\lambda_a(1) = a_d - a(1)$, in this context the bar means integration of the Parisi function $a(x)$ over the interval $[0, 1]$. By direct inspection one can check that the eigenvalues verify the property $\lambda_{a*b}(x) = \lambda_a(x)\lambda_b(x)$, where $a*b(x)$ is the product of Parisi algebra

$$(a * b)_d = a_d b_d - \int_0^1 a(y)b(y)\mathrm{d}y = a_d b_d - \overline{ab} \tag{14}$$

$$(a * b)(x) = (a_d - \bar{a})b(x) + (b_d - \bar{b})a(x)$$
$$- \int_0^x (a(x) - a(y))(b(x) - b(y))\mathrm{d}y. \tag{15}$$

It can be easily checked that the $n \times n$ matrix **C** that projects on the vector of constant coordinates (*i.e.* $\mathbf{C}_{ab} = 1 \; \forall ab$) has all zero eigenvalues in the limit $n \to 0$. As a consequence the set of eigenvalues ceases to determine the matrix A in a unique way. To recover the function $(a_d, a(x))$ we need further information *e.g.* $a(0)$. Using $a(0)$ and the relation $x\dot{a}(x) = -\dot{\lambda}_a(x)$ (the dot means derivative with respect to x) which follows from the expressions (12, 13) we obtain the following inversion relations

$$a(x) = a(0) - \int_0^x \frac{\mathrm{d}y}{y}\dot{\lambda}_a = a(0)$$
$$+ \frac{\lambda_a(0) - \lambda_a(x)}{x} + \int_0^x \frac{\mathrm{d}y}{y^2}(\lambda_a(0) - \lambda_a(y)) \tag{16}$$

$$a_d = \lambda_a(1) + a(1) = a(0) + \lambda_a(0) - \int_0^1 \frac{\mathrm{d}y}{y^2}(\lambda_a(y) - \lambda_a(0)). \tag{17}$$

Given a function $f[a] = \sum_k f_k a^k$ we want to compute $f[A]$ for a generic ultrametric matrix A. The eigenvalues $\lambda_{f[A]}$ of $f[A]$ are readily obtained

$$\lambda_{f[A]}(x) = f[\lambda_a(x)]. \tag{18}$$

$$q_s(x) = \begin{cases} q_H & \text{for} & 0 \le x \le \frac{1}{\beta_1+\beta_2}y_u(q_H) \\ q_u((\beta_1+\beta_2)x) & \text{for} & \frac{1}{\beta_1+\beta_2}y_u(q_H) \le x \le \frac{1}{\beta_1+\beta_2}y_u(p_d) \\ p_d & \text{for} & \frac{1}{\beta_1+\beta_2}y_u(p_d) \le x \le \frac{1}{\beta_s}y_u(p_d) \\ q_u(\beta_s x) & \text{for} & \frac{1}{\beta_s}y_u(p_d) \le x \le \frac{1}{\beta_s}y_u(q_{EA}(\beta_s)) \\ q_{EA}(\beta_s) & \text{for} & \frac{1}{\beta_s}y_u(q_{EA}(\beta_s)) \le x \le 1 \end{cases} \tag{32}$$

$$p(x) = \begin{cases} q_H & \text{for} & 0 \le x \le \frac{1}{\beta_1+\beta_2}y_u(q_H) \\ q_u((\beta_1+\beta_2)x) & \text{for} & \frac{1}{\beta_1+\beta_2}y_u(q_H) \le x \le \frac{1}{\beta_1+\beta_2}y_u(p_d) \\ p_d & \text{for} & \frac{1}{\beta_1+\beta_2}y_u(p_d) \le x \le 1. \end{cases} \tag{33}$$

To use the formulas (16, 17) we need to know $f[A](0)$. The expression for a product is

$$\begin{aligned} (ab)(0) &= (a_d - \bar{a})b(0) + (b_d - \bar{b})a(0) \\ &= \lambda_b(0)a(0) + \lambda_a(0)b(0). \end{aligned} \tag{19}$$

As a consequence we have the following expression for the powers of A

$$\begin{aligned} (a^{n+1})(0) &= \lambda_a(0)(a^n)(0) + \lambda_a(0)^n a(0) \longrightarrow (a^n)(0) \\ &= n(a_d - \bar{a})^{n-1}a(0), \end{aligned} \tag{20}$$

therefore

$$\begin{aligned} f[A](0) &= \sum_k f_k(a^k)(0) = \sum_k f_k k(a_d - \bar{a})^{k-1}a(0) \\ &= a(0)\frac{\mathrm{d}f}{\mathrm{d}a}[a_d - \bar{a}]. \end{aligned} \tag{21}$$

Summing up, the expression of a generic function $f[A]$ is

$$f[A](x) = a(0)\frac{\mathrm{d}f}{\mathrm{d}a}[a_d - \bar{a}] + \int_0^x \mathrm{d}y\, \dot{a}\frac{\mathrm{d}f}{\mathrm{d}a}[\lambda_a(y)] \tag{22}$$

$$\begin{aligned} \frac{1}{n}\mathrm{Tr}f[A] &= a(0)\frac{\mathrm{d}f}{\mathrm{d}a}[a_d - \bar{a}] + f[a_d - \bar{a}] \\ &\quad - \int_0^1 \frac{f[\lambda_a(y)] - f[a_d - \bar{a}]}{y^2}\mathrm{d}y. \end{aligned} \tag{23}$$

The generalization to a function $g[A_1, \dots, A_p]$ of p Parisi matrices is straightforward

$$\begin{aligned} g[A_1, \dots, A_p](x) &= \sum_{i=1}^p a_i(0)\frac{\partial g}{\partial a_i}[a_{1d} - \bar{a}_1, \dots, a_{pd} - \bar{a}_p] \\ &\quad + \int_0^x \mathrm{d}y \sum_{i=1}^p \dot{a}_i \frac{\partial g}{\partial a_i}[\lambda_{A_1}, \dots, \lambda_{A_p}] \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{1}{n}\mathrm{Tr}g[A_1, \dots A_p] &= \sum_{i=1}^p a_i(0)\frac{\partial g}{\partial a_i}[a_{1d} - \bar{a}_1, \dots, a_{pd} - \bar{a}_p] \\ &\quad + g[a_{1d} - \bar{a}_1, \dots, a_{pd} - \bar{a}_p] \\ &\quad - \int_0^1 \frac{g[\lambda_{A_1}, \dots, \lambda_{A_p}](y) - g[a_{1d} - \bar{a}_1, \dots, a_{pd} - \bar{a}_p]}{y^2}\mathrm{d}y. \end{aligned} \tag{25}$$

Appendix B

In this appendix we check that the solutions (6, 7) verify the SP equation. We consider the general situation where a non zero magnetic field is present. The Free energy functional read

$$\begin{aligned} F_n = &-\beta_1^2 \sum_{ab}(f(Q_{1ab}) + H^2 Q_{1ab}) - \beta_2^2 \sum_{ab}(f(Q_{2ab}) + H^2 Q_{2ab}) \\ &- 2\beta_1\beta_2 \sum_{ab}(f(P_{ab}) + H^2 P_{ab}) - \mathrm{Tr}\ln\hat{Q} - n. \end{aligned} \tag{26}$$

The dependence on $\hat{Q}$ can be simplified trough $\mathrm{Tr}\ln\hat{Q} = \mathrm{Tr}\ln(Q_1 Q_2 - P^2)$. The SP equations then read

$$\beta_1^2 f'(Q_{1\ ab}) + \beta_1^2 H^2 = -\left(\frac{1}{\hat{Q}}\right)_{1ab} = \frac{-Q_2}{Q_1 Q_2 - P^2} \tag{27}$$

$$\beta_1\beta_2 f'(P_{ab}) + \beta_1\beta_2 H^2 = -\left(\frac{1}{\hat{Q}}\right)_{12ab} = \frac{P}{Q_1 Q_2 - P^2}. \tag{28}$$

We recall the definition of the temperature-independent function $y_u(q)$ and of its inverse $q_u(y)$

$$y_u(q) = \frac{f'''(q)}{2(f''(q))^{\frac{3}{2}}}. \tag{29}$$

The self-overlap $q_{EA}(\beta)$ of the equilibrium states at a certain temperature is given by

$$q_d - q_{EA} = \frac{1}{\beta\sqrt{f''(q_{EA})}}. \tag{30}$$

The minimum overlap q_H in presence of a magnetic field H is independent of the temperature and is given by

$$H^2 = q_H f''(q_H) - f'(q_H). \tag{31}$$

In presence of a magnetic field the generalization of the solutions (6, 7) is

See equations (32, 33) above.

$$\left(\frac{-Q_2}{Q_1 Q_2 - P^2}\right)(x) = \frac{(q_{2d} - \bar{q}_2)^2 q_1(0) + (p_d - \bar{p})^2 q_2(0) - 2(q_{2d} - \bar{q}_2)(p_d - \bar{p})p(0)}{((q_{2d} - \bar{q}_2)(q_{1d} - \bar{q}_1) - (p_d - \bar{p})^2)^2}$$
$$+ \int_0^x dy \frac{\lambda_{q_2}^2 \dot{q}_1 + \lambda_p^2 \dot{q}_2 - 2\lambda_{q_2}\lambda_p \dot{p}}{(\lambda_{q_1}\lambda_{q_2} - \lambda_p^2)^2} \tag{34}$$

$$\left(\frac{P}{Q_1 Q_2 - P^2}\right)(x) = -\int_0^x dy \frac{\lambda_{q_2}\lambda_p \dot{q}_1 + \lambda_p \lambda_{q_1} \dot{q}_2 - \lambda_p^2 \dot{p} - \lambda_{q_2}\lambda_{q_1}\dot{p}}{(\lambda_{q_1}\lambda_{q_2} - \lambda_p^2)^2}$$

$$- \frac{(p_d - \bar{p})(q_{1d} - \bar{q}_1)q_2(0) + (p_d - \bar{p})(q_{2d} - \bar{q}_2)q_1(0) - (p_d - \bar{p})^2 p(0) - (q_{1d} - \bar{q}_1)(q_{2d} - \bar{q}_2)p(0)}{((q_{2d} - \bar{q}_2)(q_{1d} - \bar{q}_1) - (p_d - \bar{p})^2)^2}. \tag{35}$$

$$\left(\frac{-Q_2}{Q_1 Q_2 - P^2}\right)(x) = \frac{(\lambda_{q_2}(q_H) - \lambda_p(q_H))^2}{(\lambda_{q_2}\lambda_{q_1}(q_H) - \lambda_p^2(q_H))^2} q_H + \int_{q_H}^{q(x)} \frac{(\lambda_{q_2}(q) - \lambda_p(q))^2}{(\lambda_{q_2}\lambda_{q_1}(q) - \lambda_p^2(q))^2} dq \tag{36}$$

$$\left(\frac{P}{Q_1 Q_2 - P^2}\right)(x) = \frac{(\lambda_p(q_H) - \lambda_{q_2}(q_H))(\lambda_p(q_H) - \lambda_{q_1}(q_H))}{(\lambda_{q_2}\lambda_{q_1}(q_H) - \lambda_p^2(q_H))^2} q_H + \int_{q_H}^{q(x)} \frac{(\lambda_p(q) - \lambda_{q_2}(q))(\lambda_p(q) - \lambda_{q_1}(q))}{(\lambda_{q_2}\lambda_{q_1}(q) - \lambda_p^2(q))^2} dq. \tag{37}$$

These solutions are valid for any couple of temperatures $T_1 \geq T_2$ both below the critical temperature and for any p_d between q_H and $q_1(1)$ which is the self-overlap of the states of the system at the higher temperature. To check that they verify the SP equations (27, 28) we need to express their r.h.s. in the Parisi form, this is readily done applying equation (24)

See equations (34, 35) above.

In the small-x region the three functions are equal so the previous expressions simplify in this region to

See equations (36, 37) above.

The various quantities entering the previous expressions read

$$\lambda_{q_1}(q) = \frac{1}{\beta_1 + \beta_2}\left(\frac{1}{\sqrt{f''(q)}} + \frac{\beta_2/\beta_1}{\sqrt{f''(p_d)}}\right) \quad \text{for } x \leq x(p_d)$$
$$\lambda_{q_1}(q) = \frac{1}{\beta_1}\frac{1}{\sqrt{f''(q)}} \quad \text{for } x \geq x(p_d)$$
$$\lambda_p(p) = \frac{1}{\beta_1 + \beta_2}\left(\frac{1}{\sqrt{f''(p)}} - \frac{1}{\sqrt{f''(p_d)}}\right) \quad \text{for } x \leq x(p_d)$$
$$\lambda_p(p) = 0 \quad \text{for } x \geq x(p_d). \tag{38}$$

When evaluating the quantities entering the integrals the dependence on p_d disappears so as the discontinuity at $x(p_d)$:

$$\frac{(\lambda_{q_2}(q) - \lambda_p(q))^2}{(\lambda_{q_2}\lambda_{q_1}(q) - \lambda_p^2(q))^2} = \beta_1^2 f''(q);$$

$$\frac{(\lambda_p(q) - \lambda_{q_2}(q))(\lambda_p(q) - \lambda_{q_1}(q))}{(\lambda_{q_2}\lambda_{q_1}(q) - \lambda_p^2(q))^2} = \beta_1\beta_2 f''(q). \tag{39}$$

Evaluating the integrals in (36, 37) through (39) we obtain

$$\frac{-Q_2}{Q_1 Q_2 - P^2}(x) = \beta_1^2 \left(f'(q_1(x))\right.$$
$$\left. + f''(q_H)q_H - f'(q_H)\right) = \beta_1^2(f'(q_1(x)) + H^2) \tag{40}$$

$$\frac{P}{Q_1 Q_2 - P^2}(x) = \beta_1\beta_2 \left(f'(p(x))\right.$$
$$\left. + f''(q_H)q_H - f'(q_H)\right) = \beta_1\beta_2(f'(p(x)) + H^2). \tag{41}$$

Therefore the SP equations are verified in the small-x region; now for $x \geq x(p_d)$ we have $\dot{p} = 0$ and $\lambda_p = 0$ so the equation for p is immediately verified while $q_1(x)$ and $q_2(x)$ decouple

$$\frac{-Q_2}{Q_1 Q_2 - P^2}(x) = \frac{-Q_2}{Q_1 Q_2 - P^2}(x(p_d)) + \int_{p_d}^{q_1(x)} \frac{dq}{\lambda_{q_1}^2} \tag{42}$$

$$= \beta_1^2 \left(f'(p_d) + f''(q_H)q_H - f'(q_H) + f'(q_1(x)) - f'(p_d)\right)$$
$$= \beta_1^2(f'(q_1(x)) + H^2). \tag{43}$$

We skip the explicit evaluation of the free energy; it turns out to be the sum of the free energies at temperature T_1 and T_2, as it should.

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